J. Phys. Soc. Jpn. **52** (1983) Suppl. p. 235–238

DYNAMICS OF A SPIN-GLASS MODEL WITH INFINITE-RANGE INTERACTIONS

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The various static susceptibilities which can be obtained for the Sherrington-Kirkpatrick model below the freezing temperature T_f are discussed. The dynamical susceptibility $\chi(\omega,T)$ is calculated for a Glauber spin glass at all temperatures. Its static limit $\omega\to o$ leads for $T \leq T_f$ to $\chi(\omega,T) \propto \omega^\nu$ with $\nu=1/2$, at least to order $(1-T/T_f)^3$.

1. Static Susceptibilities

Conventional spin glasses are dilute magnetic alloys in which the magnetic atoms are randomly distributed over the lattice sites. Edwards and Anderson [1] were the first who considered a spin glass model with random bonds instead of random sites. Their model was further simplified by Sherrington and Kirkpatrick (SK) [2] by assuming infinite-range interactions between all spins. We consider an Ising model

$$H = -\frac{1}{2}\sum_{ij} J_{ij}\, S_i S_j - \sum_i h_i\, S_i \quad , \qquad S_i = \pm 1 \quad , \tag{1}$$

with the magnetic fields h_i (in units of μ_B) in which all spins interact with the same Gaussian bond distribution

$$P(J_{ij}) = (N/2\pi \tilde{J}^2)^{1/2} \exp\,[-\,NJ_{ij}^2/2\,\tilde{J}^2]\ . \tag{2}$$

Here, N is the number of spins and the scaling of the variance as N^{-1} ensures a sensible thermodynamic limit.

The model (1) with (2) has not yet been solved exactly. The first reliable calculation of a static susceptibility based on replica symmetry breaking due to Parisi [3] yielded $\chi(T)=T_f^{-1}$ (in units of $(g\mu_B)^2$) at all temperatures below T_f. This result has also been obtained by Sompolinsky [4] and Hertz [5].

However, $\chi=T_f^{-1}$ is not the only stable static susceptibility of the SK model. The equations of Thouless, Anderson and Palmer (TAP) [6] have a number of solutions below T_f which increases exponentially with N [7,8]. These solutions can be envisaged as local minima of the free energy in the configuration space which for $N\to\infty$ are separated by infinitely high energy barriers. Depending on the average over these minima, one can define infinitely many static susceptibilities which are all stable. The systems is non-ergodic [9]. As a consequence, the time-averaged spin glass parameter [1]

$$\tilde{q}(T) = \lim_{t\to\infty}\lim_{N\to\infty}\ [\,<S_i(o)S_i(t)>]_J \tag{3}$$

is different from the ensemble average

$$q(T) = \lim_{h\to\infty}\lim_{N\to\infty}\ [\,<S_i>^2]_J \tag{4}$$

where $<...>$ means thermal averaging and $[...]_J$ averaging over the bond distribution (2).

The parameter q(T) is non-zero only if a symmetry breaking field is applied. It is determined in general by all local minima or by the total phase space, weighted with Boltzmann factors. The evaluation of the partition function of small systems, extrapolated to $N\to\infty$, yields $1-q \propto T$ for $T \ll T_f$.

In contrast, the parameter $\tilde{q}(T)$ is obtained from Monte Carlo (MC) simulations [10] which by construction yield the time average and in which the system remains for $T\to 0$ in a single (representative) minimum with $1-\tilde{q} \propto T^2$ and with the susceptibility $\tilde{\chi}(T)=\beta(1-\tilde{q}) \propto T$ where $\beta=T^{-1}$. This low temperature result of MC simulations agrees with the solution obtained by TAP [6].

The different spin glass parameters are connected to different local dynamical susceptibilities $\chi_{ii}(\omega)$ and $\tilde{\chi}_{ii}(\omega)$. The fluctuation-dissipation theorem reads

$$\int_{-\infty}^{\infty} dt\, e^{i\omega t} [<S_i(o)S_i(t)>-<S_i>^2]_J=(2T/\omega)\,\mathrm{Im}\,\chi_{ii}(\omega)$$
$$=(2T/\omega)\,\mathrm{Im}\,\tilde{\chi}_{ii}(\omega)+2\pi\Delta\delta(\omega) \tag{5}$$

where $\tilde{\chi}_{ii}(\omega)$ is regular for $\omega\to o$. Equation (5) leads for $\omega\to o$ to

$$\Delta = \tilde{q} - q \quad , \tag{6}$$

explicitely exhibiting Δ as a measure of the non-ergodicity. It also indicates that the non-ergodicity is effective only in the static limit. Applying Kramers-Kronig relations to (5) yields the local static susceptibilities

$$\chi_{ii}(\omega=o)=\pi^{-1}\int_{-\infty}^{\infty}d\omega'\ \mathrm{Im}\,\chi_{ii}(\omega')/\omega' = \beta(1-q)= \beta(1-\tilde{q}+\Delta) \quad , \tag{7}$$

$$\tilde{\chi}_{ii}(\omega=o) = \beta(1-\tilde{q}). \tag{8}$$

The result (7) is surprizing since Parisi obtained for the total susceptibility [3]

$$\chi_{tot}= N^{-1} \sum_{ij} \chi_{ij} = \beta(1-\int_o^1 dx\ q(x)\ ) \tag{9}$$

introducing a continuous set of spin glass parameters q(x). Eqs. (7) and (8) imply $\chi_{ij}\neq 0$ for $i\neq j$, in contrast to a symmetry argument proposed earlier [11]. As pointed out in [9], off-diagonal terms give a finite contribution to χ_{tot} for the following reasons: (1) there are N(N-1) off-diagonal terms in contrast to N diagonal ones, and (2) the limit $h\to o$ does not lead to the state with h=o since the system occupies in both cases different states which are separated by infinitely high energy barriers. The difference between χ_{ii} and χ_{tot} is not clear in the papers of Parisi and Sompolinsky [3,4]. Hertz [5] recently proved that indeed $\chi_{tot}=T_f^{-1}$ whereas $\chi_{ii}\neq T_f^{-1}$.

Depending on time, magnetic field and history (field cooled or isothermal magnetization) the system can go into rather different sets of local minima. Since part of the potential barriers becomes infinitely high below T_f, the system usually remains blocked in these states. A classification of a set of spin glass parameters $\tilde{q}(x) = [<S_i(o)S_i(t_x)>]_J$ in terms of a distribution of relaxation times t_x, all of which become infinite in the thermodynamic limit, has been introduced by Sompolinsky [4]. Here, $t_x \propto \exp[\beta E_x]$ is the relaxation time for surmounting the energy barrier $E_x\to\infty$ with the continuous variable $x\in[0,1]$ and $\tilde{q}(x=o)=o$ for the longest time x=o. Sompolinsky identifies the maximum value $\tilde{q}(1)=\tilde{q}$ with the time average (3) and determines $\tilde{q}(T)$ by the de Almeida-Thouless (AT) stability criterion [12]. The non-ergodicity on the time scale x is described by a second set of parameter $\Delta(x)$ with the maximum value $\Delta(o)=\Delta$ for the largest time (the "true" equilibrium) where Δ is defined in (7). Near to the AT stability line or for the shortest time x=1 the system becomes ergodic with $\Delta(1)=0$, and the local and total susceptibilities become equal. The identification $\tilde{q}(1)=\tilde{q}$ does not seem to be necessary. It leads to a susceptibility $\tilde{\chi}_{ii}(T)=\tilde{\chi}_{tot}=\beta(1-\tilde{q})$ which is initially flat just below T_f whereas Hertz [5] derives for $\tilde{\chi}_{ii}$ an expression which is symmetric above T_f, in agreement with the measured a.c. susceptibility. Hertz [5] also stresses a necessary relation between $\Delta(x)$ and $\tilde{q}(x)$. The free energy functional which leads to $\tilde{q}(x,T)$ and $\Delta(x,T)$ introduced by Sompolinsky recently has been derived by de Dominicis et al. [13] via a replica method. Sommers [14] showed, that $\tilde{q}(o)$ becomes finite in a non-zero field h with $\tilde{q}(o) \propto h^{2/3}$ for all temperatures $0< T< T_f$ and small fields.

2. Dynamical Susceptibility

The Ising model has no inherent spin dynamics. Here we consider the Glauber model in which the spins are coupled to a "heath bath" which induces spontaneous spin flips and can be identified with the conduction electrons of the host. We follow closely the classical paper of Suzuki and Kubo [15] and extend a theory for $T > T_f$ of Kinzel and the author [16] to $T< T_f$. We have in mean field approximation (which is exact for the infinite range model)

$$(1+\tau\, d/dt)\ \langle S_i(t)\rangle = \tanh \beta \langle h_i^{eff}\rangle \quad , \tag{10}$$

with the Korringa relaxation time τ and where the effective field h_i^{eff} at the lattice site i is given by the TAP equations [6]

$$h_i^{eff} = h_i + \Sigma\, J_{ij}\, S_j - \beta\, \tilde{J}^2(1-\tilde{q})\, S_i \ . \tag{11}$$

We diagonalize the random symmetric matrix

$$J_{ij} = J_{ji} = \sum_{\lambda=1}^{N} J_\lambda \langle i/\lambda\rangle\langle \lambda/j\rangle \tag{12}$$

with the real orthonormal eigenvectors $\langle\lambda/i\rangle$. This leads with

$$q_\lambda = \sum_i \langle\lambda/i\rangle\langle S_i\rangle \quad , \quad h_\lambda = \sum_i \langle\lambda/i\rangle h_i \tag{13}$$

to

$$(1+\tau\, d/dt)q_\lambda = \sum_i \langle i/\lambda\rangle \tanh\left\{\sum_{\lambda'} \left[(\beta J_{\lambda'} - \beta^2\tilde{J}^2(1-\tilde{q}))q_{\lambda'} + \beta h_{\lambda'}\right]\langle i/\lambda'\rangle\right\} \tag{14}$$

For $T > T_f$, $h_i\to o$ and small deviations out of equilibrium equs. (10) and (14) can be linearized since $\langle S_i\rangle_{h=o}=0$ in thermal equilibrium. For $T\leq T_f$ the modes λ are strongly coupled. As shown by Sompolinsky [17] this can be taken into account by a self-energy $\chi_o(\omega)$ with $\chi_o(\omega)=1$ for $T \geq T_f$. One has for the staggered dynamical susceptibility

$$\chi_\lambda(\omega) = dq_\lambda/dh_\lambda = \beta[\chi_o^{-1}(1-i\omega\tau) - \beta J_\lambda + \beta^2\tilde{J}^2(1-\tilde{q})]^{-1} \ . \tag{15}$$

With the eigenvalue density of a random matrix

$$\rho(J) = (2\pi\tilde{J}^2)^{-1}(4\tilde{J}^2-J^2)^{1/2} \qquad (|J|\leq 2\tilde{J}) \tag{16}$$

the total susceptibility is easily found to be ($\bar{T}\equiv T/T_f$)

$$\chi(\omega,T) = N^{-1}\sum_{ij}\chi_{ij}(\omega,T) = N^{-1}\sum_\lambda \chi_\lambda(\omega,T)$$

$$= \frac{\beta}{2}\{\bar{T}^2\chi_o^{-1}(1-i\omega\tau) + 1-\tilde{q} - ([\bar{T}^2\chi_o^{-1}(1-i\omega\tau) + 1-\tilde{q}]^2 - 4\bar{T}^2)^{1/2}\} \ . \tag{17}$$

The limit $\lim_{\omega\to o}\chi(\omega)$ can be chosen in such a way that it agrees with the TAP solution $\tilde{\chi}=\beta(1-\tilde{q})$. This is the susceptibility for the shortest (infinite) relaxation time $t_{x=1}$ and yields

$$\chi_o(\omega=o) = 1-\tilde{q} \ . \tag{18}$$

The condition (18) is identical with the onset of non-ergodic behaviour [17]. Inserting $\tilde{q}(T)=\bar{t}-\bar{t}^2+\bar{t}^3-+\ldots$ with $\bar{t}=1-\bar{T}$ from the AT stability line [4] one has at least to order $\bar{t}^3$ the limit $\lim_{\omega\to o}\chi(\omega,T)\propto \omega^{1/2}$ $(T\leq T_f)$. This $\omega^{1/2}$ law indicates a marginal phase transition: The staggered susceptibility (15) diverges for all $T \leq T_f$ and the spin correlation function $[<S_i(o)S_i(t)>]_J$ decays in the long-time limit as $(t/\tau)^{-1/2}$, in agreement with the Monte Carlo data of Kirkpatrick and Sherrington [10]. The same critical exponent $\nu=\frac{1}{2}$ has been obtained for the infinite-range vector model [18]. However, our result differs from $\nu(T)=1/2-\pi^{-1}t+O(t^2)$ derived by Sompolinsky and Zippelius [19] for a soft spin model. Possibly, this discrepancy is due to the different models.

One can also consider the limit $\omega \to o$ in which $\chi(\omega,T)$ reduces to the susceptibility $\chi(T)=\beta(1-\tilde{q}+\Delta(o))$ [4] for the longest (infinite) relaxation time $t_{x=o}$. This leads to a relation between $\chi_o(\omega=o)$, Δ and $\tilde{q}$ which replaces (18).

3. References

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